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Abstract. Multi-fuzzy set theory is an extension of fuzzy set theory, which deals with the multi dimensional fuzziness. It was introduced by S. Sebastian and T.V. Ramakrishnan in 2010. Thenceforth, the theory is developing through many angles. This paper proposes the notion of distance measure of multi-fuzzy sets. Furthermore, the paper explores some results related to this concepts.

INTRODUCTION

The theory of multi-fuzzy sets was introduced in terms of multi dimensional membership functions. Multi-fuzzy set was introduced by S. Sebastian and T.V. Ramakrishnan in 2010. They explored some notions of multi-fuzzy sets related to set theory, group theory, topology and logic. It is noticed that multi- fuzzy set is an extension of fuzzy sets, *L*-fuzzy sets and intuitionistic fuzzy sets. This article endeavours to contribute some fruitful results related to multi-fuzzy sets. This paper aims to propose the concept of distance measure of multi-fuzzy sets. Some basic results related to it are explored.

The rest of the paper will be organized as follows: Section 2 gives a quick look at fuzzy sets and its distance measure. It also describes the basic concepts related to multi-fuzzy sets. Section 3 proposes the concept of distance measure of multi-fuzzy sets and investigates its properties. Section 4 concludes the article.

PRELIMINARIES

In this section, necessary backgrounds are provided. Initially the concept of fuzzy sets and distance measure between fuzzy sets are given. Then, the notion of multi-fuzzy sets and standard operations on it are discussed.

Fuzzy sets

A fuzzy set is determined by a membership function which accepts all the intermediate values between zero and one. The values of the membership function, called membership degrees or grades of membership, precisely specify to what extent an element belongs to a fuzzy set, i.e., to the concept it belongs.

Fuzzy sets [1] Let X be a given universal set, which is always a crisp set. A fuzzy set A on X is characterized by a function $A : X \to [0,1]$, called fuzzy membership function, which assigns to each object a grade of membership ranging between zero and one. A fuzzy set A is defined as

$$A = \{(x, A(x)); x \in X\}$$

where $A(x)$ is the fuzzy membership value of x in X.

Each fuzzy set is completely determined by its membership function. Depending on the universal set, the membership functions are either discrete or continuous. It is important to note that the definition of the membership degrees is subjective and context dependent, meaning that each person has his own perception of each concept and that the interpretation is dependent on the universe of discourse and context inwhich the fuzzy set is used.

The collection of all fuzzy sets on the universal set X is denoted by $\boldsymbol{FS}(X)$. A fuzzy set is considered to be empty if the membership degrees of all the elements of the universe are equal to zero.

Standard Operations on Fuzzy sets

The three basic operations on crisp sets, viz. complement, intersection, and union can be generalized to fuzzy sets in more than one way. One particular generalization, usually reffered to as standard fuzzy set operations, has a special

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significance in fuzzy set theory.

Complement: [1] Let A be a fuzzy set on the universal set X. The standard complement, denoted as $\overline{A}$, of A with respect to X is defined for all $x \in X$ by the equation

$$\overline{A}(x) = 1 - A(x)$$

Elements of X for which $A(x) = \overline{A}(x)$ are called equilibrium points of A.

Intersection: [1] Let A and B be two fuzzy sets on the universal set X. Their standard intersection, denoted as $A \cap B$, is defined for all $x \in X$ by the equation

$$(A \cap B)(x) = \min\{A(x), B(x)\}$$

where 'min' denotes the minimum operator.

Union: [1] Let A and B be two fuzzy sets on the universal set X. Their standard union, denoted as $A \cup B$, is defined for all $x \in X$ by the equation

$$(A \cup B)(x) = \max\{A(x), B(x)\}$$

where 'max' denotes the maximum operator.

Distance Measure of Fuzzy Sets

A distance measure of two fuzzy sets is a measure that describes the difference between fuzzy sets. In literature, many different axiomatic and non-axiomatic definitions of distance measures between fuzzy sets have been introduced. Most of the axiomatic definitions have been introduced independently from each other. They have been proposed in different contexts, with different purposes and using different nomenclatures. In 2013 Couso et al. [2] reviewed and organized some remarkable definitions of distance measures of fuzzy sets, including all of them in a common framework, and analyzed and displayed their formal relationships. They have laid bare some close relations between some apparently different axiomatic definitions. The distance measure of fuzzy sets is discussed in [3, 4]. Xuecheng [5] gave an axiom definition for distance measure between fuzzy sets as follows:

Distance Measure of Fuzzy Sets [5] Let $\boldsymbol{P}(X)$ be the collection of all crisp subsets of X. A real function $d : \boldsymbol{FS}(X) \times \boldsymbol{FS}(X) \to \mathbb{R}^+$ is called a distance measure, if d has the following properties:

1. $d(A,B) = d(B,A)$ for all $A,B \in \boldsymbol{FS}(X)$.
2. $d(A,A) = 0$ for all $A \in \boldsymbol{FS}(X)$.
3. $d(C,\bar{C}) = \max_{A,B \in \boldsymbol{FS}(X)} d(A,B)$ for all $C \in \boldsymbol{P}(X)$.
4. For all $A,B,C \in \boldsymbol{FS}(X)$, if $A \subseteq B \subseteq C$, then $d(A,B) \leq d(A,C)$ and $d(B,C) \leq d(A,C)$.

Multi-fuzzy sets

Multi fuzzy sets are defined in terms of ordered sequences of membership functions.

Multi-fuzzy sets [6] Let X be a non empty set and let $\{L_i; i \in \mathbb{N}\}$ be a family of complete lattices where $\mathbb{N}$ is the set of all positive integers. A multi fuzzy set A in X is a set of ordered sequences

$$A = \{\langle x, \mu_1(x), \mu_2(x), ...\rangle; x \in X\}$$

where $\mu_i \in L_i^X$ for $i \in \mathbb{N}$. The function $\mu_A = (\mu_1, \mu_2, ...)$ is called a multi membership function of multi fuzzy set A.

If the sequences of the membership function have only k terms, k is called dimension of A. Let $L_i = [0,1]$ for $i = 1,2,...,k$, then the set of all multi fuzzy sets in X of dimension k is denoted by $M^kFS(X)$

Standard Operations on Multi-fuzzy sets

Let $\{L_i : i \in \mathbb{N}\}$ be a family of complete lattices and

$$A = \{\langle x, \mu_1(x), ..., \mu_i(x), ...\rangle; x \in X,\ \mu_i \in L_i^X,\ i \in \mathbb{N}\}$$
$$B = \{\langle x, \nu_1(x), ..., \nu_i(x), ...\rangle; x \in X,\ \nu_i \in L_i^X,\ i \in \mathbb{N}\}$$

be multi-fuzzy sets in a nonempty set X. The following relations and operations are defined on multi fuzzy sets [**?**]:

1. **Subset**: A is a subset of B, denoted by $A \subseteq B$, if and only if $\mu_i(x) \leq \nu_i(x)$ for every $x \in X$ and $i \in \mathbb{N}$.
2. **Equality**: A is equal to B, denoted by $A = B$, if and only if $\mu_i(x) = \nu_i(x)$ for every $x \in X$ and $i \in \mathbb{N}$.
3. **Union**: Union of A and B, denoted by $A \cup B$, is defined by

$$A \cup B = \{\langle x, \mu_1(x) \vee \nu_1(x), , ..., \mu_i(x) \vee \nu_i(x), ...\rangle; x \in X,\ \mu_i \in L_i^X,\ i \in \mathbb{N}\}$$

4. **Intersection**: Intersection of A and B, denoted by $A \cap B$, is defined by

$$A \cap B = \{\langle x, \mu_1(x) \wedge \nu_1(x), , ..., \mu_i(x) \wedge \nu_i(x), ...\rangle; x \in X,\ \mu_i \in L_i^X,\ i \in \mathbb{N}\}$$

DISTANCE MEASURE OF MULTI-FUZZY SETS

In this section, we propose the idea of distance measure of multi-fuzzy sets. Consider $M^kFS(X)$, the set of all multi fuzzy sets in X of dimension k. Let $\xi_k(X)$ be the collection of all k tuples whose entries are either zero or one.
Distance Measure of Multi-fuzzy sets: A real function $\mathscr{D} : M^kFS(X) \times M^kFS(X) \to \mathbb{R}^+$ is called a distance measure of multi-fuzzy sets, if $\mathscr{D}$ satisfies the following axioms;

1. $\mathscr{D}(\mathscr{A},\mathscr{B}) = \mathscr{D}(\mathscr{B},\mathscr{A})$ for every $\mathscr{A},\mathscr{B} \in M^kFS(X)$.
2. $\mathscr{D}(\mathscr{A},\mathscr{A}) = 0$ for every $\mathscr{A} \in M^kFS(X)$.
3. $\mathscr{D}(\mathscr{C},\bar{\mathscr{C}}) = \max\limits_{\mathscr{A},\mathscr{B} \in \boldsymbol{MS}_{(n,k)}(X)} \mathscr{D}(\mathscr{A},\mathscr{B})$ for every $\mathscr{C} \in \xi_k(X)$.
4. For every $\mathscr{A},\mathscr{B},\mathscr{C} \in M^kFS(X)$, if $\mathscr{A} \subseteq \mathscr{B} \subseteq \mathscr{C}$, then $\mathscr{D}(\mathscr{A},\mathscr{B}) \leq \mathscr{D}(\mathscr{A},\mathscr{C})$ and $\mathscr{D}(\mathscr{B},\mathscr{C}) \leq \mathscr{D}(\mathscr{A},\mathscr{C})$

Normal Distance Measure A distance measure $\mathscr{D}$ on $M^kFS(X)$ is said to be normal on $M^kFS(X)$, if $\max\limits_{\mathscr{A},\mathscr{B} \in \boldsymbol{MS}_{(n,k)}(X)} \mathscr{D}(\mathscr{A},\mathscr{B}) = 1$.
Proposition: Let $\mathscr{D}$ be a distance measure on $M^kFS(X)$. Then

$$\hat{\mathscr{D}}(\mathscr{A},\mathscr{B}) = \frac{\mathscr{D}(\mathscr{A},\mathscr{B})}{\max\limits_{\mathscr{G},\mathscr{H} \in M^kFS(X)} \mathscr{D}(\mathscr{G},\mathscr{H})}$$

for every $\mathscr{A},\mathscr{B} \in M^kFS(X)$, is a normal distance measure on $M^kFS(X)$.

Let d be any distance measure of fuzzy sets. For multi-fuzzy sets $\mathscr{A}$ and $\mathscr{B}$ in $M^kFS(X)$, define

$$\mathscr{D}(\mathscr{A},\mathscr{B}) = \min_{j \in \mathbb{N}_k} d(A_i^j, B_i^j) \tag{1}$$

Theorem: $\mathscr{D}(\mathscr{A},\mathscr{B})$, given in equation 1, is a distance measure between the multi-fuzzy sets $\mathscr{A}$ and $\mathscr{B}$ in X.
Proof Axioms (1) and (2) are obvious.
Axiom(3): Let $\mathscr{C}$ be any multi-fuzzy set in $M^kFS(X)$. Clearly,

$$\mathscr{D}(\mathscr{C},\overline{\mathscr{C}}) \leq \max_{\mathscr{A},\mathscr{B} \in M^kFS(X)} \mathscr{D}(\mathscr{A},\mathscr{B}) \tag{2}$$

Now, for any multi-fuzzy sets $\mathscr{A},\mathscr{B} \in M^kFS(X)$,

$$d(C_j,\overline{C}_j) \geq d(A_j,B_j)$$

for every $j \in \mathbb{N}_k$. Therefore,

$$\min_{j\in\mathbb{N}_k} d(C_j,\overline{C}_j) \geq \min_{j\in\mathbb{N}_k} d(A_j,B_j)$$

Then, $\mathscr{D}(\mathscr{C},\overline{\mathscr{C}}) \geq \mathscr{D}(\mathscr{A},\mathscr{B})$ for every $\mathscr{A},\mathscr{B} \in M^kFS(X)$. Therefore,

$$\mathscr{D}(\mathscr{C},\overline{\mathscr{C}}) \geq \max_{\mathscr{A},\mathscr{B}\in M^kFS(X)} \mathscr{D}(\mathscr{A},\mathscr{B}) \tag{3}$$

Combining inequalities (**??**key23) and (**??**key24), it follows that

$$\mathscr{D}(\mathscr{C},\overline{\mathscr{C}}) = \max_{\mathscr{A},\mathscr{B}\in M^kFS(X)} \mathscr{D}(\mathscr{A},\mathscr{B})$$

Axiom(4): Suppose $\mathscr{A},\mathscr{B}$ and $\mathscr{C}$ are multi-fuzzy sets in $M^kFS(X)$ such that $\mathscr{A} \subseteq \mathscr{B} \subseteq \mathscr{C}$. Then $A_j \subseteq B_j \subseteq C_j$ for every $j \in \mathbb{N}_k$. Then, from the definition for distance measure of fuzzy sets,

$$d(A_j,B_j) \geq d(A_j,C_j)$$

for every $j \in \mathbb{N}_k$. Therefore,

$$\min_{j\in\mathbb{N}_k} d(A_j,B_j) \geq \min_{j\in\mathbb{N}_k} d(A_j,C_j)$$

and hence $\mathscr{D}(\mathscr{A},\mathscr{B}) \geq \mathscr{D}(\mathscr{A},\mathscr{C})$. Analogously, $\mathscr{D}$ satisfies second inequality also. Therefore, $\mathscr{D}(\mathscr{A},\mathscr{B})$ satisfies all the axioms. Thus $\mathscr{D}(\mathscr{A},\mathscr{B})$ is a distance measure between the multi-fuzzy sets $\mathscr{A}$ and $\mathscr{B}$ in X.

Using the properties of fuzzy distance measures and definition of distance measures of multi-fuzzy sets, the following theorems can be proved easily:

Theorem: Let $\mathscr{A}$ and $\mathscr{B}$ be multi-fuzzy sets in $M^kFS(X)$. Suppose fuzzy distance measure d satisfies the property $A = B \Leftrightarrow d(A,B) = 0$. Then, $\mathscr{A} = \mathscr{B} \Leftrightarrow \mathscr{D}(\mathscr{A},\mathscr{B}) = 0$.

Proof: Suppose $\mathscr{A} = \mathscr{B}$. Then $A_j = B_j$ where A_j and B_j are fuzzy sets for $1 \leq j \leq k$. That is, if and only if $d(A_j,B_j) = 0$ for $1 \leq j \leq k$. That is, if and only if

$$\min_{j\in\mathbb{N}_k} d(A_j,B_j) = 0$$

That is, $\mathscr{D}(\mathscr{A},\mathscr{B}) = 0$.

Theorem: Let $\mathscr{A}$ and $\mathscr{B}$ be multi-fuzzy sets in $M^kFS(X)$. Suppose fuzzy distance measure d satisfies the property $A \cap B = \phi \Leftrightarrow d(A,B) = 1$. Then, $\mathscr{A} \cap \mathscr{B} = \Phi \Leftrightarrow \mathscr{D}(\mathscr{A},\mathscr{B}) = 1$

Proof: Suppose $\mathscr{A} \cap \mathscr{B} = \Phi$. Then $A_j \cap B_j = \phi$ where A_j and B_j are fuzzy sets for $1 \leq j \leq k$. That is, if and only if $d(A_j,B_j) = 1$ for $1 \leq j \leq k$. That is, if and only if

$$\min_{j\in\mathbb{N}_k} d(A_j,B_j) = 1$$

That is, $\mathscr{D}(\mathscr{A},\mathscr{B}) = 1$.

Theorem: Let $\mathscr{A}$ and $\mathscr{B}$ be multi-fuzzy sets in $M^kFS(X)$. Let $\mathscr{M}$ be the multi-fuzzy set over X such that membership value for each $x \in X$ is the k-tuple whose entries equal to 0.5. Then, $\mathscr{A} = \mathscr{M} \Rightarrow \mathscr{D}(\mathscr{A},\bar{\mathscr{A}}) = 0$.

Proof: Suppose $\mathscr{A} = \mathscr{M}$. Then $A_j = M$ where A_j is a fuzzy set for $1 \leq j \leq k$ and M is a fuzzy set whose membership value for each element is 0.5. Then $\overline{A_j} = A_j$, and therefore, $d(A_j,\overline{A_j}) = d(A_j,A_j) = 0$ for $1 \leq j \leq k$. Then

$$\min_{j\in\mathbb{N}_k} d(A_j,\overline{A_j}) = 0$$

That is, $\mathscr{D}(\mathscr{A},\bar{\mathscr{A}}) = 0$.

Theorem: Let $\mathscr{A}$ and $\mathscr{B}$ be multi-fuzzy sets in $M^kFS(X)$. Let $\xi_k(X)$ be the collection of all k tuples whose entries are either zero or one. Suppose fuzzy distance measure d satisfies the property $d(A,\bar{A}) = 1 \Leftrightarrow A = I$ or $A = 0$ and $0 \leq d(A,B) \leq 1$. Then, $\mathscr{D}(\mathscr{A},\bar{\mathscr{A}}) = 1 \Leftrightarrow \mathscr{A} \in \xi_k(X)$.

Proof: Suppose $\mathscr{D}(\mathscr{A},\bar{\mathscr{A}}) = 1$. Then

$$\min_{j\in\mathbb{N}_k} d(A_j,\overline{A_j}) = 1$$

Then $d(A_j,\overline{A_j}) = 1$ for $1 \leq j \leq k$. That is, if and only if $A_j = I$ or $A_j = 0$. That is, $\mathscr{A} \in \xi_k(X)$.

CONCLUSION

In this article, the concept of distance measuire of multi-fuzzy sets is introduced. Moreover, some elementary properties of this concept are established.

REFERENCES

1. L. Zadeh, "Fuzzy sets," Information and Control **8**, 338–353 (1965).
2. L. S. I. Couso, L. Garrido, "Similarity and dissimilarity measures between fuzzy sets: A formal relational study," Information Sciences **299**, 122–141 (2013).
3. M. Luo and Z. Cheng, "The distance between fuzzy sets in fuzzy metric spaces," International Conference on Fuzzy Systems and Knowledge Discovery **128**, 197–201 (2015).
4. A. ROSENFELD, "Distances between fuzzy sets," Pattern Recognition Letters **3**, 229–233 (1985).
5. L. Xuecheng, "Entropy, distance measure and similarity measure of fuzzy sets and their relations," Fuzzy Sets and Systems **52**, 305–318 (1992).
6. S. Sebastian and T. Ramakrishnan, "Multi-fuzzy sets: An extension of fuzzy sets," Fuzzy Information and Engineering **3**, 35–43 (2011).